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THE CONJUGACY PROBLEM FOR KNOT GROUPS

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It is a deep theorem of Waldhausen [14] that the group $\Gamma = \pi_1(S^3 \backslash K)$ of any tame knot K has a solvable word problem. Even the powerful methods of Waldhausen's approach, using sufficiently large irreducible 3-manifolds, do not shed light on the conjugacy problem.

Using small cancellation theory, Weinbaum, Appel and Schupp [9] solved the problem for alternating knots, Abbud [1] for 3-braid links, and the automatic groups theory developed in [4] solved it for hyperbolic knots.

Combining topological constructions of Haken, Jaco and Oertel [7], algebraic properties of hyperbolic groups given by Gromov, and mostly Thurston's geometrization and Dehn-Surgery theorems ([11], [13]), we present a complete solution of the conjugacy problem for knot groups.

In Section 1 we cut the knot complement into locally homogeneous pieces and study their nature. Section 2 is devoted to the conjugacy problem in amalgamated products and its reduction to corresponding questions in each of the components. These questions are treated in Section 3 for the hyperbolic case and in Section 4 for the Seifert fibered case.

Although we deal only with knot groups, our methods seem to generalize to link groups using the results of Burde and Murasugi [2] and to all 3-manifolds satisfying Thurston geometrization conjucture using [12].

1. SPLITTING THE KNOT COMPLEMENT

Given a Wirtinger presentation of a knot group $\Gamma = \pi_1(S^3 \backslash K)$ or a description of the knot via braids, we can easily construct a triangulation of the knot complement $M = S^3 \backslash K$. M is an irreducible 3-manifold with boundary, so it is a Haken manifold. For these we have the following splitting theorem due to Jaco and Shalen:

Theorem 1.1 [8]. *Let M be a Haken 3-manifold. Then there exists a 2-sided, incompressible 2-manifold $W \subset M$, unique up to ambient isotopy, having the following three properties*:

(i) *The components of W are annuli and tori, and none of them are boundary parallel in M.*

(ii) *Each connected component of $M \backslash W$ is either a Seifert pair or a simple pair.*

(iii) *W is minimal with respect to inclusion among all two-sided 2-manifolds in M having properties* (i) *and* (ii).

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By Thurston's geometrization theorem for Haken manifolds, all simple pairs in a knot complement are (finite-volume) hyperbolic, and Seifert fibered pairs are classified by the following.

PROPOSITION 1.2 ([8], VI 3.4). *A Seifert fibered 3-manifold that is contained with incompressible boundary in a knot complement is either a torus knot space, a cable space or a composing space.*

Remark. A composing space is a compact 3-manifold homeomorphic to $W \times S^1$, where W is a disk with holes.

In order to treat the conjugacy problem for knot groups, we would like to reduce the question of conjugacy in the ambient manifold to certain questions in each of the geometric pieces. Therefore, our first step will be to cut the manifold into geometric pieces and classify them. To find the Jaco–Shalen canonical splitting, we need to review the theory of fundamental surfaces according to Haken, Jaco and Oertel [6], [7].

Given a Heegard splitting (e.g. the one inherited from a triangulation) of a manifold, there exists a system of linear Diophantine equations $\mathscr{L}$ so that any normal surface with respect to the given Heegard splitting corresponds to a solution of the system $\mathscr{L}$. Haken has obseved that there is a (unique) finite set of non-negative solutions to $\mathscr{L}$ such that every non-negative solution is a non-negative integral linear combination of this finite set of solutions. These are called fundamental solutions of the normal equations $\mathscr{L}$. If a fundamental solution is realizable (i.e. corresponds to a surface), then the resulting normal surface is called a fundamental surface. Since one is able to find the fundamental solutions of $\mathscr{L}$ the set of fundamental surfaces is constructible. Furthermore, each normal surface is an integral linear combination of the finite set of fundamental surfaces.

PROPOSITION 1.3. *Let M be a non-trivial knot complement. If M contains a properly embedded annulus which is not boundary parallel, then for any handlebody-decomposition of M one of the fundamental solutions of the normal equations corresponds to such an annulus.*

Proof. There is no compressing disk or a properly embedded Möbius band in a non-trivial knot complement. Therefore, an incompressible annulus can not be obtained by a connected sum of several fundamental surfaces such that all the elements in the sum are not annuli. □

If a fundamental surface is an annulus, we check if it belongs to the Jaco–Shalen splitting 2-manifold by requiring that its boundaries be meridians and each of the components it separates not be solid tori. Having all such incompressible annuli at hand we look for incompressible tori among the fundamental surfaces. By the same arguments as in Proposition 1.3, if there exists an incompressible torus which is not boundary parallel, there is a fundamental one. We look for the innermost incompressible torus and, once we have done this, we repeat the entire process (i.e. look for the annuli and then for tori).

PROPOSITION 1.4. *There is an algorithm to decide if a submanifold of a prime knot complement with boundary components composed of tori and containing no essential torus is hyperbolic or Seifert fibered.*

Proof. We simply look for an essential annulus. If there exists one it has to separate our manifold into two solid tori (in case we have a torus knot), or a solid torus and $T^2 \times I$ (in case the knot is cabled). Alternatively, one can perform a surgery with high enough coefficients on the boundaries of our submanifold and check the hyperbolicity of the obtained manifold, as is done in [12]. □

2. AMALGAMATED PRODUCTS

In case the complement of our knot is not hyperbolic or Seifert fibered, its fundamental group splits as an amalgamated product over Z or $Z \oplus Z$. To deal with the conjugacy problem in that case, we need to discuss some basic properties of words in amalgamated products.

Definition 2.1. An element $\gamma \in \Gamma = A *_C B$ is called cyclically reduced if γ has a reduced form $\gamma = \gamma_1 \dots \gamma_r$, where γ_1 and γ_r are not in the same factor.

THEOREM 2.2 ([10], 4.6). *Let $\Gamma = A *_C B$. Then every element of Γ is conjugate to a cyclically reduced element of Γ. Moreover, suppose that γ is a cyclically reduced element of Γ. Then the following hold.*

(i) *If γ is conjugate to an element $c \in C$, then γ is in some factor and there is a sequence $c, c_1, \dots, c_t, \gamma$ where $c_i \in C$ and consecutive terms of the sequence are conjugate in a factor.*
(ii) *If γ is conjugate to an element γ' in some factor, but not conjugate to an element of C, then γ and γ' are in the same factor and are conjugate in that factor.*
(iii) *If γ is conjugate to an element $p_1 \dots p_r$, where $r \geq 2$, and p_i, p_{i+1} as well as p_1, p_r are in distinct factors, then γ can be obtained by cyclically permuting $p_1, \dots, p_r$ and then conjugating by an element of C.*

PROPOSITION 2.3. *Let $\gamma \in \Gamma = \pi_1(S^3 \backslash K)$ and suppose $\Gamma = A *_C B$ where C is either infinite cyclic or $Z \oplus Z$. Then there exists an effective algorithm to find a cyclically reduced form for γ.*

Proof. We need to decide if elements $a \in A$ or $b \in B$ belong to C or not. In case $C = Z \oplus Z$, C is its own centralizer and since the word problem is decidable for a knot group we just check if γ commutes with the generators of C. If C is infinite cyclic we do the same for the boundary torus whose fundamental group contains C as a subgroup and then check if a given element in the given torus belongs to the cyclic subgroup C. □

If the two elements in question $\gamma_1, \gamma_2 \in \Gamma$ have conjugates in either A or B we need to solve the conjugacy problem in these components. Otherwise we are in case (iii) of Theorem 2.2, so we need to decide if there exists a conjugating element $c \in C$, i.e. an element such that (up to permutation of the cyclically reduced forms) $c\gamma_1 c^{-1} = \gamma_2$. Let $\gamma_1 = a_1 b_1 \dots a_n b_n$, $\gamma_2 = e_1 f_1 \dots e_n f_n$. We have: $\gamma_2 = c\gamma_1 c^{-1} = (ca_1 c^{-1})(cb_1 c^{-1}) \dots (cb_n c^{-1})$. In particular, the two sides have the same reduced form, so $e_1 = ca_1 c^{-1}\tilde{c}$ for some $\tilde{c} \in C$, or equivalently $e_1 \in Ca_1 C$.

By splitting the knot complement, we may assume that A is the fundamental group of a geometric piece. Therefore, in the following sections we have to treat the above decision problem in order to indicate the possibilities for the conjugating elements $c \in C$, together with the conjugacy problem for the geometric pieces themselves. We start with the hyperbolic case and continue with the Seifert fibered one.

3. FINITE VOLUME HYPERBOLIC MANIFOLDS

Throughout this section we will assume our knot group $\Gamma = \pi_1(S^3 \setminus K) = A *_C B$, where A is the fundamental group of a finite volume hyperbolic manifold. We start by proving the decidability of the conjugacy problem for the fundamental groups of such hyperbolic manifolds, although these groups are bi-automatic and so the decidability follows from [4]. We give a different proof since it serves as a motivation for our general approach.

THEOREM 3.1. *Let M be a finite volume hyperbolic manifold and let $G = \pi_1(M)$. The conjugacy problem for G is decidable.*

Proof. Given a finite volume hyperbolic manifold M, one can produce a sequence of negatively curved manifolds $\{N_s\}_{s=1}^{\infty}$ converging to M in the Gromov topology by performing surgeries on M with increasing surgery coefficients [13]. The fundamental groups of these surgered manifolds $\{\pi_1(N_s)\}_{s=1}^{\infty}$ converge to $\pi_1(M)$ both algebraically and geometrically [3].

Let g_1, g_2 be given elements in $\pi_1(M)$. On the one hand we look for an element $g \in \pi_1(M)$ such that $g_1 = g g_2 g^{-1}$. On the other hand we check if the projections of g_1 and g_2 are conjugate in $\pi_1(N_s)$. We do that by first applying Gromov's isoperimetric inequality to find a δ for which $\pi_1(N_s)$ with generating system inherited from G is δ-hyperbolic ([5], 2). Then we check if the two projections are conjugate in $\pi_1(N_s)$ by applying Gromov's solution to the conjugacy problem for hyperbolic groups ([5], 7).

Suppose g_1 is not conjugate to a boundary element. Then there exists a closed geodesic in the free homotopy class of g_1. If g_2 is not conjugate to g_1 then for s large enough the geodesics in the free homotopy classes of g_1 and g_2 in N_s are different, and so our procedure terminates. If both g_1 and g_2 are conjugate to boundary elements on the same boundary component, g_1 to $m^{r_1}l^{s_1}$ and g_2 to $m^{r_2}l^{s_2}$, verifying conjugacy in the manifolds N_{s_1} and N_{s_2} with distinct boundary slopes (p_1, q_1), (p_2, q_2) gives:

$$(r_1, s_1) = (r_2, s_2) + \lambda(p_1, q_1)$$

$$(r_1, s_1) = (r_2, s_2) + \mu(p_2, q_2).$$

So $\lambda(p_1, q_1) = \mu(p_2, q_2)$ which implies $\lambda = \mu = 0$ and, therefore, $(r_1, s_1) = (r_2, s_2)$. □

PROPOSITION 3.2. *Let M be a finite volume hyperbolic manifold, $G = \pi_1(M)$, and $C < G$ the fundamental group of a boundary torus. Let $a_1, a_2 \in G \setminus C$ and suppose $a_1 \in C a_2 C$. Then there are unique $c_1, c_2 \in C$ such that $a_1 = c_1 a_2 c_2$.*

Proof. Let $a_1 = c_1 a_2 c_2 = \tilde{c}_1 a_2 \tilde{c}_2$. Then $a_2^{-1} \tilde{c}_1^{-1} c_1 a_2 = \tilde{c}_2 c_2^{-1}$. Elements in C belong to distinct conjugacy classes so a_2 is in the centralizer of $\tilde{c}_1^{-1} c_1$, but $a_2 \notin C$, and we have $\tilde{c}_1^{-1} c_1 = \tilde{c}_2 c_2^{-1} = 1$. □

LEMMA 3.3. *Let H be a hyperbolic group, h_1 and h_2 elements of infinite order with cyclic normalizer in H, such that either h_1 and h_2 commute or no power of h_1 is conjugate to a power of h_2. Let $a_1, a_2 \in H$, $a_2 \notin \langle h_1 \rangle$. There exists an effective algorithm to decide if $a_1 \in \langle h_1 \rangle a_2 \langle h_2 \rangle$.*

Proof. By the same arguments we have used in Proposition 3.2, if a_1 belongs to the double coset of a_2 there exist unique n_1, n_2 such that $a_1 = h_1^{n_1} a_2 h_2^{n_2}$. Powers of an infinite order element in a hyperbolic group form a quasi-geodesic ([5], 7). This behavior of powers guarantees the existence of an integer p for which:

$$|h_i^{2p}| - 200\delta \geq |h_i^p|$$

which implies by Lemma 7.2C of [5]:

$$\delta^{-1}|h_i^p||h_i^{pu}| \geq u|h_i^p|$$

$$|h_i^{pu}| \geq u\delta$$

for an arbitrary integer u. These inequalities imply ([5], 7) that the powers h_i^{pu} remain within a distance $d = 100\delta(1 + \log(10\delta|h_i^p|))$ from the corresponding geodesic.

Having the quasi-geodesics coefficients, we look for effective bounds on the powers n_1, n_2. We scan the powers (h_i^{pu}) until we get to a power u_1 for which:

$$|h_1^{p \cdot u_1}a_2h_2^q| \geq 4d + |a_1| + 2p(|h_1| + |h_2|) + 10\delta$$

for all integers q which satisfy:

$$|q| \leq (|h_1^{u_1 p}| + 4\delta + |a_1| + 2p(|h_1| + |h_2|) + 10\delta)\delta^{-1}p$$

The uniqueness of the couple n_1, n_2 guarantees the existence of such u_1, which gives us bounds on (n_1, n_2):

$$|n_1| \leq pu_1$$

$$|n_2| \leq (|h_1^{pu_1}| + 4d + |a_1| + 2p(|h_1| + |h_2|) + 10\delta)\cdot\delta^{-1}\cdot p$$

THEOREM 3.4. *Let M be a finite volume hyperbolic manifold, $G = \pi_1(M)$, C the fundamental group of a boundary component, and $g_1, g_2 \in G$; $g_2 \notin C$. Then there exists an effective algorithm to decide if $g_1 \in Cg_2C$.*

Proof. We look at two particular closed manifolds, N_{s_1} and N_{s_2}, obtained by surgery on the original manifold M with surgery coefficients (p_1, q_1), (p_2, q_2) on the boundary torus having the fundamental group C. In the manifolds N_{s_1} and N_{s_2}, C projects to cyclic groups $\langle\gamma_1\rangle$, $\langle\gamma_2\rangle$.

By Lemma 3.3 we can find (if there exist at all) k_1, k_2, k_3, k_4 for which:

$$v_1(g_1) = \gamma_1^{k_1}v_1(g_2)\gamma_1^{k_2}$$

$$v_2(g_1) = \gamma_2^{k_3}v_2(g_2)\gamma_2^{k_4}$$

where v_i are the natural homomorphisms from G onto $\pi_1(N_{s_i})$. If there exists $c_1, c_2 \in C$ such that $g_1 = c_1g_2c_2$ then:

$$c_1 = (r_1, s_1) = (\chi_1, \chi_2) + \lambda(p_1, q_1)$$

$$c_1 = (r_1, s_1) = (\tilde{\chi}_1, \tilde{\chi}_2) + \mu(p_2, q_2)$$

These two equations uniquely determine c_1. Applying the same procedure for c_2, we get the only candidates for the couple c_1, c_2. □

4. SEIFERT FIBERED MANIFOLDS

The only Seifert fibered components appearing as geometric pieces in a knot complement are cable spaces or the complement of a torus knot [8]. We start by solving the conjugacy problem in these components.

THEOREM 4.1. *Let* $\Gamma = \langle x, y | x^p = y^q \rangle$ *or* $\Gamma = \langle x, y, z | xy = yx;\ x^p y^q = z^q \rangle$ *where* $(p, q) = 1, |p| > 1, |q| > 1$. *The conjugacy problem for* Γ *is decidable.*

Proof. In both cases Γ has the short exact sequence $1 \to Z \to \Gamma \to G \to 1$, where G is a hyperbolic group. Given elements $u, v \in \Gamma$ we use the Gromov solution for the conjugacy problem in hyperbolic groups to decide if the projections $\hat{u}, \hat{v} \in G$ are conjugate ([5], 7). If there exists $\hat{t} \in G$ such that $\hat{t}\hat{u}\hat{t}^{-1} = \hat{v}$, we pick a representative $t \in \Gamma$ which projects to $\hat{t} \in G$ and check if $tut^{-1} = v$. Clearly the validity of this equality does not depend on the particular t we've chosen, since the possible t differ by elements which commute with u. □

Solving the conjugacy problem in a Seifert fibered component, we need to solve it for $\Gamma = A *_C B$, where A is the fundamental group of a Seifert fibered component and C is either Z or $Z \oplus Z$. Given two elements $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 = a_1 b_1 \ldots a_n b_n$ and $\gamma_2 = e_1 f_1 \ldots e_n f_n$; $a_i, e_i \in A$; $b_i, f_i \in B$, we start by finding all $c, \tilde{c} \in C$ for which $e_1 = c a_1 c^{-1} \tilde{c}$. Let $w \in C$ be a generator of the center of A, and let $h \in C$ satisfy $\langle h, w \rangle = C$. Clearly, if $e_1 = h^{\alpha_1} w^{\beta_1} a_1 w^{-\beta_1} h^{-\alpha_1} h^{\alpha_2} w^{\beta_2}$, then $e_1 = h^{\alpha_1} a_1 h^{-\alpha_1} h^{\alpha_2} w^{\beta_2}$. Let v denote the natural homomorphism v: $A \to A/\langle w \rangle = G$. G is a hyperbolic group and in G we must have:

$$v(e_1) = v(h)^{\alpha_1} v(a_1) v(h)^{-\alpha_1} v(h)^{\alpha_2}. \tag{4.1}$$

Even though the group G has torsion, $v(h)$ is of infinite order and its normalizer is the cyclic subgroup generated by $v(h)$, so by Lemma 3.3 there exists an effective algorithm to find the unique α_1, α_2 for which equality (4.1) holds. This couple also defines β_2 uniquely and we have:

$$e_1 = h^{\alpha_1} a_1 h^{-\alpha_1} h^{\alpha_2} w^{\beta_2} = h^{\alpha_1} a_1 h^{-\alpha_1} \tilde{c}. \tag{4.2}$$

In order to determine the possible $c \in C$ for which $c\gamma_1 c^{-1} = \gamma_2$ we still need to find the w-exponent β_1. We have: $\gamma_2 = e_1 f_1 \ldots e_n f_n = c\gamma_1 c^{-1} = c a_1 b_1 \ldots a_n b_n c^{-1} = c a_1 c^{-1} \tilde{c} c \tilde{c}^{-1} b_1 \ldots a_n b_n c^{-1} = e_1 c \tilde{c}^{-1} b_1 \ldots a_n b_n c^{-1}$. In particular, $c\tilde{c}^{-1} b_1 = f_1 \hat{c}$. Suppose first, B is the fundamental group of a geometric component. If B is the fundamental group of a finite volume hyperbolic manifold, we apply the same argument that we have used in Lemmas 3.3, 3.4 for double cosets to find a possible β_1. If B is the fundamental group of a Seifert fibered manifold, we take the quotient of B modulo its center which is a hyperbolic group. Since w is an element of infinite order, we can apply Lemma 3.3 to find the possible β_1.

If B splits as an amalgamated product $B = B_1 *_T B_2$ and $c\tilde{c}^{-1} b_1$ or $f_1 \hat{c}$ are of length bigger than 1 (otherwise we can reduce the problem to a unique geometric piece), we apply the following. If $w \notin B_1$ we get a bound on β_1 in terms of the lengths of $\tilde{c}^{-1} b_1$ and f_1 in the amalgamated product $B_1 *_T B_2$, which leave us with a finite number of possibilities for c. If $w \in B_1$ we need to decide if $c\tilde{c}^{-1} \overline{b_1} = \overline{f_1} t$ where $\overline{b_1}, \overline{f_1} \in B_1, t \in T$ which is similar to our previous decision problem (Lemmas 3.3 and 3.4 remain valid for double cosets with respect to different boundary groups) so again we have a unique possible β_1.

Having β_1 in hand, we have determined the possible c completely, so we just need to check if $\gamma_2 = c\gamma_1 c^{-1}$, which can be done since we have a solution to the word problem. Thus, we have completed the main result of our paper:

THEOREM 4.2. *The conjugacy problem for knot groups is decidable.*

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